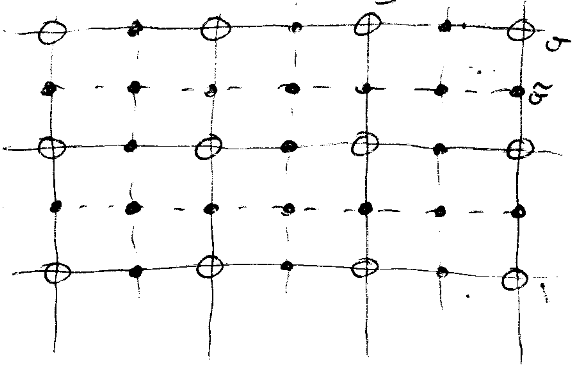
The Continuum Limit

Now we have a complete prescription for calculating physical (ie. gauge-invariant) quantities in lattice gauge theory. A mathematician would say we have "reduced the problem to quadrature". But have we solved QCD? No! All ~~our~~ results are still expressed in terms of the lattice spacing a. Recalling the first lecture, we expect our predictions to become insensitive to the cutoff (ie. dimensionless ratios become independent of a) only in the vicinity of a continuous phase transition. Where is the transition for QCD?

Back to the Ising Model:



Define subsets of spins

σ [on lattice of spacing $2a$]

& $\tilde{\sigma}$ [on remaining sites]

Near a phase transition it is helpful to consider the effect of a "block-spin" transformation, ie to sum over only the spins $\tilde{\sigma}$ to leave an effective (blocked) Hamiltonian written in terms of σ.

$$\text{ie} \qquad \exp\left(-H_{2a}[\sigma]\right) \propto \sum_{\{\tilde{\sigma}\}} \exp -H_a[\sigma,\tilde{\sigma}]$$

Note we have absorbed the temperature into H; ie we write

$$H = -K\sum \sigma_i\sigma_j \qquad \text{with} \qquad K = \frac{J}{kT}$$

The block-spin transformation is often dignified by the term <u>real space renormalisation group transformation</u>, though sadly it can only be performed approximately for $d>1$.

However, it turns out that near the transition it is possible to parametrise the change in H as $a \to 2a$ by altering just a few parameters, eg. the effective temperature

$$K_a \longrightarrow K'_{2a} \left(\simeq K_a^2 \quad \text{for } K \text{ small}\right)$$

We call the set of such interactions which can be tuned to change the cutoff scale <u>relevant</u> parameters

$\Longleftrightarrow$ renormalisable interactions in QFT

eg. particle mass $M \rightarrow 2M$
correlation length $\xi \rightarrow \xi/2$

Some interactions turn out to be <u>irrelevant</u>, ie tend rapidly to zero under blocking eg. 4-spin interaction $\begin{matrix}\uparrow & \uparrow \\ \downarrow & \uparrow\end{matrix}$

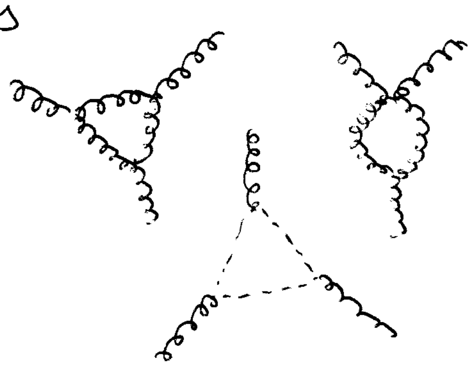
$\Longleftrightarrow$ non-renormalisable interactions

<u>Fixed points</u> of the blocking,
ie $H \rightarrow H \equiv H^*$
$K \rightarrow K \equiv K^*$ imply either $\xi = 0$ or $\xi = \infty$

The latter case is a scale-invariant point, at which the cutoff scale can be altered without changing the physics.

Now, often the response of a QFT to a change in cutoff scale can be calculated in perturbation theory

eg. Yang-Mills theory:
renormalisation of 3-pt. function



$$\Rightarrow \quad B(g) \equiv -a \frac{\partial g(a)}{\partial a} = -\frac{g^3}{16\pi^2} \cdot \frac{11N}{3}$$

(This is not quite the same as the β-fn. of standard perturbation theory, which considers g_R as a function of some physical scale μ: here we treat bare g as a function of a
The two definitions coincide to 2-loops)

$$\Rightarrow \quad g \searrow 0 \quad \text{as} \quad a \searrow 0 \quad \text{ie } B(g=0) = 0$$

$\Rightarrow$ the continuum limit lies in the perturbative régime.

Now consider the calculation of a physical mass M

eg. $M_{glueball}$, $\sqrt{K}$ in lattice QCD.

By dimensional analysis M must be some multiple of $1/a$ (the only scale in the model) and also a function of the coupling g.

But if M is physical, it must also be cut-off independent:

$$\Rightarrow \quad -a\frac{d}{da}M(a,g) = M + B(g)\frac{\partial M}{\partial g} = 0$$

to lowest order $-B(g) = \beta_1 g^3 + \beta_2 g^5 + \ldots$ with $\beta_1 = \frac{11N}{48\pi^2}$

The differential eqn. is solved by

$$M = \frac{C}{a}\exp\left(\frac{-1}{2\beta_1 g^2}\right) + \text{h.o.}$$

ie. is essentially singular as $g, a \to 0$

$\Rightarrow$ No mass is computable in perturbation theory, since it is non-analytic in g.

$\Rightarrow$ All the "physics" is in the integration constant C

ie ratios of particle masses are pure numbers independent of g & a

$$M_i = C_i \Lambda_{QCD}$$

Λ_{QCD} is the number which tells us how "strong" the strong interaction is. It has dimensions of mass. Its numerical value depends strongly on details of regularisation.

eg. for SU(3) Yang-Mills $\quad \frac{\Lambda_{\overline{MS}}}{\Lambda_{latt}} = 28.8$

Current estimate for $\Lambda_{\overline{MS}} \simeq 200\text{ MeV}$

So, to get predictions from the lattice:

(i) determine a in fm. by calculating $\sqrt{K}$ or $M_{glueball}$ in lattice units, using various values of the bare coupling g (ie β). This is the "callibration measurement" $M_0 a$.

(ii) Any further mass calculations yield predictions, ie of dimensionless ratios M_i / M_0. With luck, if we are close enough to the continuum limit universality will hold, ie.

$$\frac{M_i}{M_0} \text{ is independent of } \beta \text{ as } \beta \nearrow \infty$$

This happy situation is known as <u>SCALING</u>

(iii) If β is sufficiently large, perturbation theory in $g^2 \sim 1/\beta$ should become accurate, and we observe scaling according to the RG β-fn: ie we have analytic control

$$\text{ie} \quad M_i a \propto \exp\left(-\frac{1}{2\beta_1 g^2}\right) \quad \text{1-loop result}$$

$$M_i a \propto \left(g^2 \beta_1\right)^{-\beta_2/2\beta_1^2} \exp\left(-\frac{1}{2\beta_1 g^2}\right)\left(1 + O(g^2)\right) \quad \text{2-loop result.}$$

This perfect situation is <u>ASYMPTOTIC SCALING</u>

Once we know that perturbation theory describes the scaling, we are "safe", since we know continuum Yang-Mills theory is renormalisable — simply a statement that predictions can be made cut-off independent.

How close are we to asymptotic scaling? Consider the <u>step-scaling function</u> $\Delta\beta(\beta)$; ie the change in β required to half the correlation length, ie double the physical lattice spacing:

$$\text{ie} \int_a^{2a} \frac{da'}{a'} = -\int_g^{g+\Delta g} \frac{dg'}{B(g')} = \frac{\sqrt{6}}{2} \int_\beta^{\beta - \Delta\beta} \frac{d\beta'}{\beta'^{3/2} B(\beta')} \quad \text{for } SU(3)$$

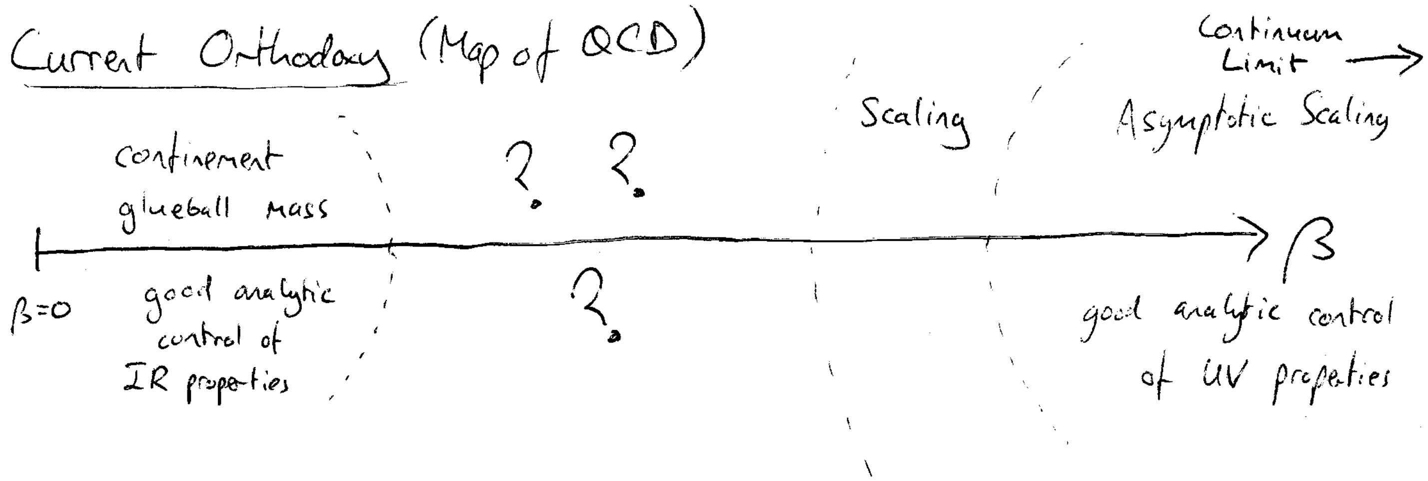
If perturbation theory holds, $B(\beta') = -\beta_1 \left(\frac{6}{\beta'}\right)^{3/2}$ to lowest order

$$\Rightarrow \quad \Delta\beta = \frac{33 \cdot 2\ln 2}{8\pi^2} = 0.579\ldots \quad \text{ie is constant}$$

Exercise: Derive the step scaling function $\Delta\beta(\beta)$ at strong coupling (hint: use the leading order expression for K)

There are asymptotically-free theories (ie. models in which we expect a similar pattern of scaling in the weak coupling limit) in $d=2$ – the $O(n)$ non-linear sigma models. Here asymptotic scaling appears not to hold even for $\xi a \sim O(100)$.

Current Orthodoxy (Map of QCD)

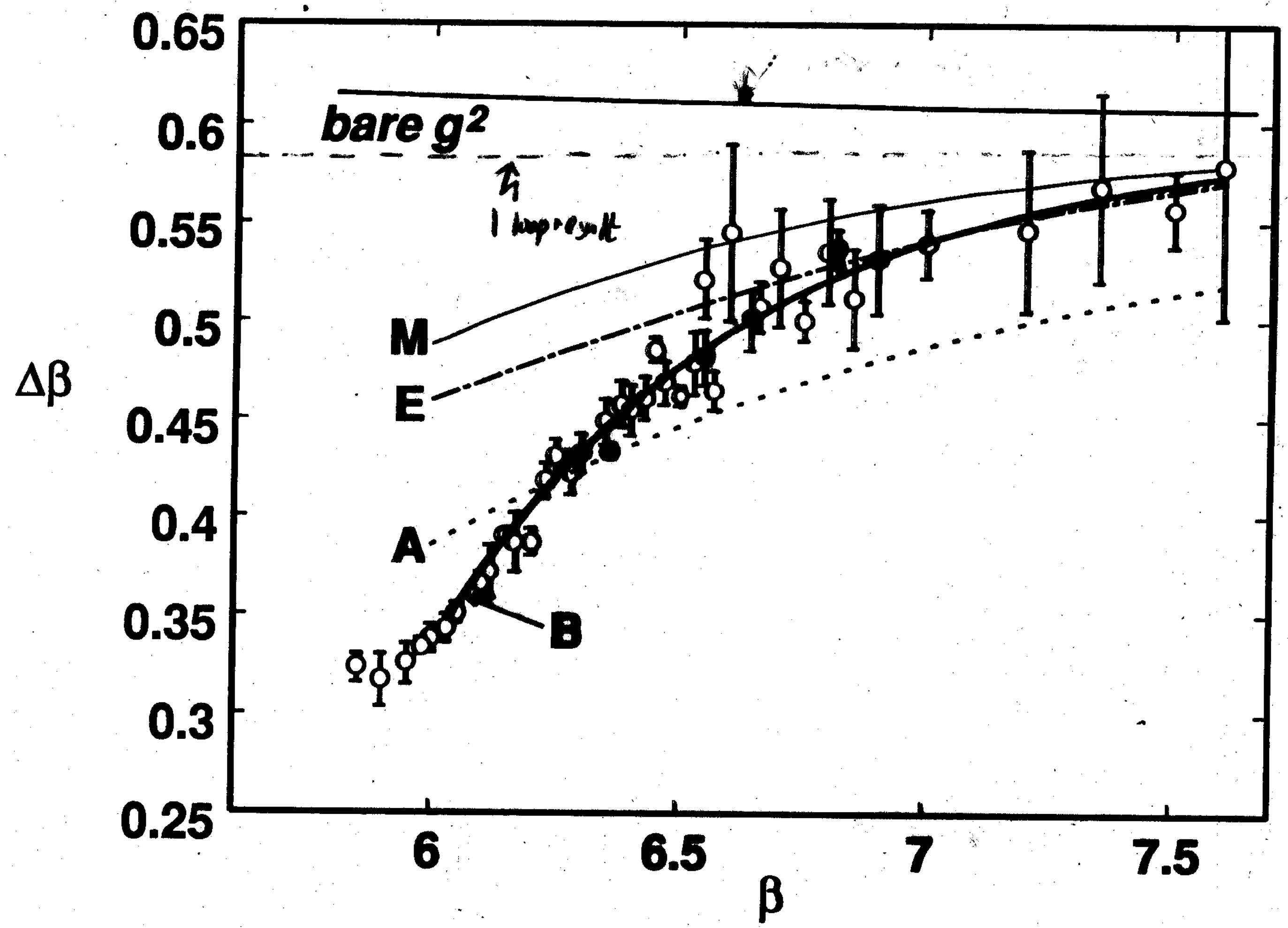


It turns out that the strong-coupling series has a non-zero radius of convergence, unlike the perturbative series, which is asymptotic. Initially, people thought there might be a region of overlap, where both methods are applicable, & hence results of one could be continued to the other $\Rightarrow$ get confinement in continuum limit.
$\Rightarrow$ analytic solution of QCD.

Unfortunately, the two regions do not overlap in practice – there is a crossover region where no analytic technique appears to hold (possibly due to a singularity in the "fundamental-adjoint" plane – ie. the extended coupling space occupied by a LGT with action
$$S_{FA} \cong -\beta_F \, \mathrm{tr}_{\underline{3}}\, U_{\mu\nu} - \beta_A \, \mathrm{tr}_{\underline{8}}\, U_{\mu\nu} \quad)$$

At least there appears to be no barrier to the continuum limit; ie a phase separation

Fig.5



Akemi et al
(QCD TARO Collab.)
1993

Was this inevitable? Perhaps not – eg. in scalar ϕ^4 theory, there is a region of overlap enabling a continuation of strong-coupling methods to continuum perturbation theory: this has enabled an 'analytic' (though not closed-form) solution to ϕ^4 theory – see Lüscher & Weisz, Nucl. Phys. B290 (1987) 25 – this strongly suggests the model is trivial, ie $\lambda_R \searrow 0$ as $a \searrow 0$. Extension to models with O(4) symmetry enables an upper bound to be placed on the Higgs mass $\sim$ 650 GeV