S Hands

Lattice Methods in Quantum Field Theory

Introduction: "Lattice field theory" is a very popular field of study at present, with its own annual conference (O(300) participants) & electronic archive (hep-lat). It is also a broad church, being at once a formal tool used in defining the path integral, a means of exploring novel theories (quantum gravity), exploring familiar ones under unfamiliar conditions of high temperature or particle density, and finally as a practical tool in obtaining QCD-related results in Standard Model phenomenology.

The great feature of the lattice method is its providing a non-perturbative regularisation for QFT, enabling systematic calculation without the aid of Feynman diagrams. Novel techniques, such as strong-coupling expansions, variational approaches (coupled cluster method) become available, as well as numerical simulation on powerful computers.

In these lectures I aim to introduce the basic ideas & some of the more important methods, and show how new ways of thinking about physics (as well as new physical results) emerge. Throughout I will focus on the path integral approach in Euclidean spacetime.

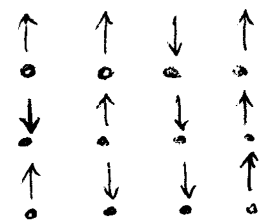
Lattice Field Theory is now a mature field (O(20) years) & there are a number of good books:

2 review articles by J.B. Kogut "Spin Systems & LGT" Rev. Mod. Phys. 51 (1979) 659
"LGT approach to QCD" Rev. Mod. Phys. 55 (1983) 775

"Quarks Gluons & Lattices" M. Creutz (Cambridge Monograph)
"Statistical Field Theory" C. Itzykson & J.-M. Drouffe (Cambridge Monograph)
"Quantum Fields on a Lattice" I. Montvay & G. Münster (— " — — " —)
"Lattice Gauge Theories" H. J. Rothe (World Scientific)
"Quantum Fields on the Computer" ed. M. Creutz (World Scientific)

Also Proceedings of the Lattice Conferences ("LAT '9x") Nucl. Phys. B [Proc. Suppl.]

Phase Transitions & Critical Phenomena



Consider a simplified model of a ferromagnet – the Ising Model

On each site of a cubic lattice lives an atomic "spin" σ_i which takes values ± 1
Assume the interaction between spins is short-ranged (Q.M. exchange)

Hamiltonian $$H[\sigma] = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - \sum_i h_i \sigma_i$$

– sign tends to align spins

denotes sum over nearest neighbour pairs

external magnetic field defined at each site

Clearly H is minimised when all spins parallel, ie. all $+1$ or all -1

Now put system in contact with a heat bath at temperature T. Thermal fluctuations allow access to states with higher energy than ground state $\rightarrow$ there are many more of these of course (higher entropy). We need to minimise not H, but the free energy $F = H - TS$

Now $F = -kT \ln Z$, with $$Z = \sum_{\{\sigma\}} \exp\left(-\frac{H[\sigma]}{kT}\right)$$ PARTITION FUNCTION

The sum $\sum_{\{\sigma\}}$ denotes the sum over all possible configurations of the spins: for a system of N spins there are 2^N terms.

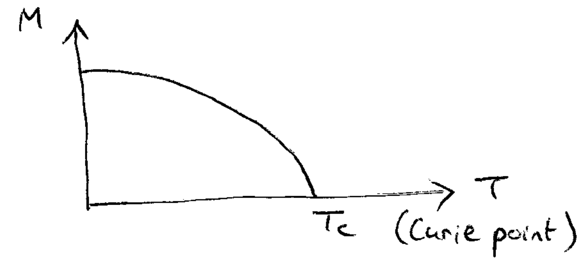
We define thermal averages, or "expectation values" via derivatives of F wrt external field h_i

eg. magnetisation $$M = \langle \sigma_i \rangle = \frac{1}{Z} \sum_{\{\sigma_i\}} \sigma_i \exp\left(-\frac{H[\sigma]}{kT}\right) = \left.\frac{-\partial F}{\partial h_i}\right|_{h=0}$$

. Because there are many more states with small M than large M, we expect $M \searrow$ as $T \nearrow$

Exercise: Show that $-kT\frac{\partial^2 F}{\partial h_i \partial h_j}$ generates the connected 2-point correlation function $\langle \sigma_i \sigma_j \rangle_c \equiv \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$

<u>Phase Transition</u>



Transition is <u>continuous</u> or <u>2^{nd} order</u> since $M(T)$ is continuous.

In the region of T_c, M & other thermodynamic quantities scale as powers of <u>reduced temperature</u> $t = \frac{T - T_c}{T_c}$

eg. magnetisation $M \propto (-t)^{\beta}$

susceptibility $\chi = \left.\frac{\partial M}{\partial h}\right|_{h=0} = \sum_j \langle \sigma_i \sigma_j \rangle_c \propto |t|^{-\gamma}$ ie. diverges as $t \to 0$

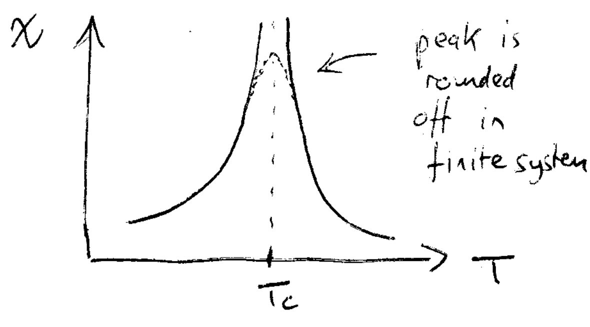
constant external field

at criticality $M|_{t=0} \propto h^{1/\delta}$

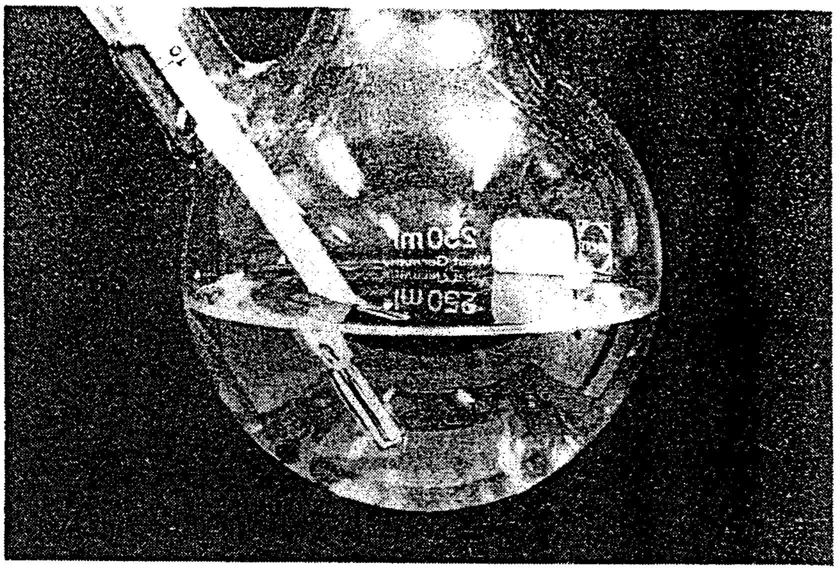
specific heat $C_v = \frac{\partial \langle H \rangle}{\partial T} \propto |t|^{-\alpha}$ diverges at criticality

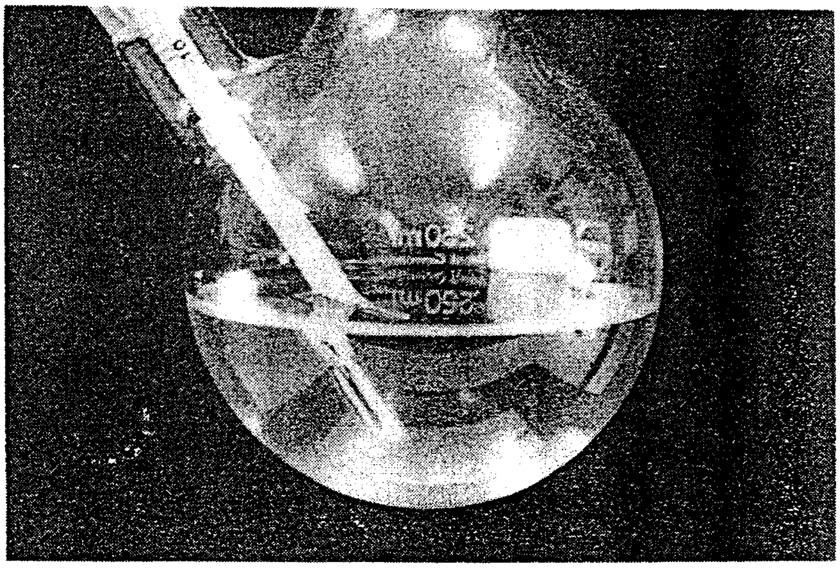
The quantities $\alpha, \beta, \gamma, \delta$ are known as <u>critical exponents</u>.
These <u>non-analyticities</u> are surprising, since Z, F & all its derivatives are defined as analytic functions of h and T. In fact, the strange scaling behaviour arises from cooperative behaviour between the <u>microscopic</u> spin degrees of freedom over <u>macroscopic</u> distances, and hence depends on the <u>thermodynamic limit</u> $N \to \infty$.

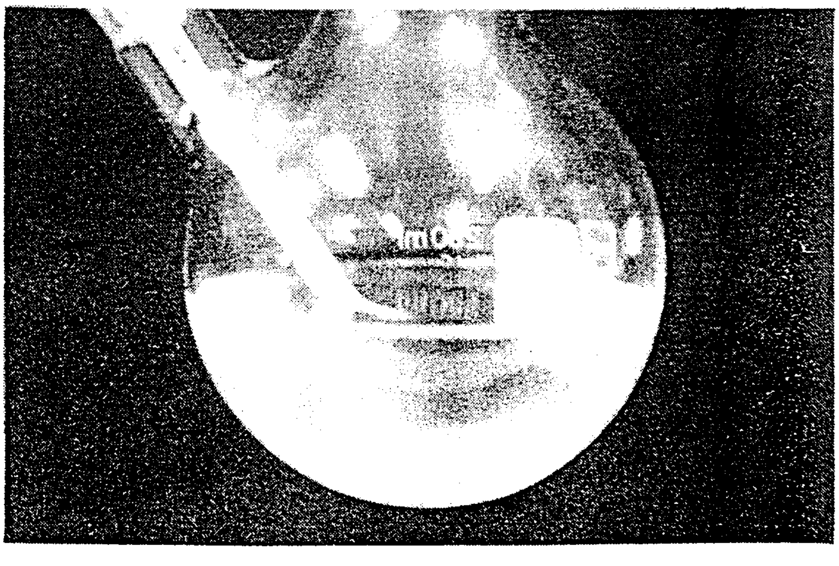
eg.



This illustration, also from the book by Binney et al, shows critical opalescence in a mixture of methanol and n-hexane, and shows the appearance of the mixture as T is lowered towards T_c.







Both statistical mechanics and QFT are systems with many degrees of freedom, displaying complicated behaviour as that number tends to ∞. In QFT this limit gives rise to UV divergences & hence the need to regularise the system, using eg. Pauli-Villars, dimensional regularisation etc.

An important concept is <u>correlation length</u> ξ, defined in terms of the asymptotic behaviour of the correlation function:

Let $|i-j| = \vec{r}$: then by translational invariance of the equilibrium state:

$$\langle \sigma_i \sigma_j \rangle_c \sim \frac{1}{r^p} \exp\left(-\frac{r}{\xi}\right) \qquad \text{for } r \gg \xi$$

$$\sim \frac{1}{r^{p'}} \qquad \text{for } r \ll \xi$$

Note $p \neq p'$ in general. As the system becomes critical $(T \to T_c)$, ξ diverges:

$$\xi \propto |t|^{-\nu}$$

At the critical point, the two-point function is pure power law.

$$\langle \sigma_i \sigma_j \rangle_c \big|_{t=0} \propto \frac{1}{r^{d-2+\eta}}$$

ν & η are two further critical exponents.

As $\xi \to \infty$, correlations over larger & larger length scales govern the system's dynamics. Once $\xi \gg a$, where a is the original lattice spacing, the system "forgets" about the original lattice, and we recover Euclidean rotational invariance in $\langle \sigma_i \sigma_j \rangle_c$; ie it becomes a function of $r = |\vec{r}|$ rather than $\vec{r}$

This leads to the important idea of <u>universality</u>; ie the critical system is completely characterised by the set of critical exponents, & is insensitive to such details as lattice type, coordination number, quantum-mechanical effects etc.

Dictionary: Statistical Mechanics $\Leftrightarrow$ Euclidean Quantum Field Theory

Dynamical Variable	σ_i	$\Leftrightarrow$	$\phi(x)$	

Hamiltonian $H[\sigma] = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - \sum_i h_i \sigma_i$

$$\Leftrightarrow \quad \text{Action } S = \int_x \frac{(\partial_\mu \phi)^2}{2} + V(\phi) + J.\phi$$

Integration measure	$\sum_{\{\sigma\}}$	$\Leftrightarrow$	$\int D\phi$	
Partition function	$Z = \sum_{\{\sigma\}} \exp\left(-\frac{H[\sigma]}{kT}\right)$	$\Leftrightarrow$	$Z = \int D\phi \exp\left(-\frac{S[\phi]}{\hbar}\right)$	Generating functional
Thermal fluctuations	kT	$\Leftrightarrow$	$\hbar$	Quantum fluctuations
external field	h	$\Leftrightarrow$	J	external source
Correlation function	$\langle \sigma_i \sigma_j \rangle$	$\Leftrightarrow$	$\langle \phi(0)\phi(x) \rangle$	Euclidean Green function
Correlation length	ξ	$\Leftrightarrow$	$1/m$	inverse mass
lattice spacing	a	$\Leftrightarrow$	$1/\Lambda$	inverse U.V. cutoff
exponent	$\eta/2$	$\Leftrightarrow$	"anomalous dimension" γ of field ϕ	
Critical region	$\xi \gg a$	$\Leftrightarrow$	field theory régime $\Lambda \gg m$	

This means that even an apparently simple model such as the Ising model can reproduce the essential features of a phase transition. Moreover, because of universality, many apparently distinct physical systems belong to the Ising 'universality class'

eg. uniaxial ferromagnets (MnF_2), liquid-vapour critical point, order-disorder in binary alloys ($CuZn$)

QCD critical point in (μ, T) plane

There are many formal similarities between the statistical model presented here, and the path integral approach to QFT. (see dictionary). Indeed, one of the most successful theoretical approaches to calculating critical exponents uses ϕ^4 scalar field theory in $4-\varepsilon$ dimensions — ϕ^4 theory falls into the Ising universality class. Conversely, we can use statistical mechanics methods in QFT. Since we don't (necessarily) believe in an underlying spacetime lattice, we want to work in a régime $\xi \gg a$ — the "field theory régime" where our predictions are independent of the fine details of the lattice structure & universality holds. In QFT language this régime is $\Lambda \gg \mu$, where μ is a physical scale (particle mass, momentum transfer) and Λ is the UV cutoff. The challenge in either case is to make physical predictions independent of Λ or a.

Exercise: Work out the "timeslice propagator" for a free scalar boson:

$$\left\langle \left(\sum_{\underline{x}} \phi(\underline{x}, 0) \right) \left(\sum_{\underline{y}} \phi(\underline{y}, t) \right) \right\rangle$$

given the momentum space propagator $\langle \phi(p)\, \phi(q) \rangle = \delta_{p,-q} \dfrac{1}{p^2+m^2}$

Show that it decays like e^{-mt}

(Use the definition of inverse F.T.: $\phi(x) = \int \frac{d^4p}{(2\pi)^4}\, \phi(p)\, e^{-ip\cdot x}$)